Some Birkhoff Interpolation Problems on the Roots of Unity

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Dedicated to Professor I. J. Schoenberg on his 80th birthday.

Submitted by Hans Schneider

ABSTRACT

The object of this note is to study three row almost Hermitian incidence matrices and to give sufficient conditions when the corresponding interpolation problem is regular on the roots of unity. In particular, a three row almost Hermitian matrix with only two nonzero entries in one row is regular on the cube roots of unity. Other situations are also examined in detail.

1. INTRODUCTION

During the last few years several papers have appeared which deal with lacunary interpolation on the roots of unity. A survey of these results is now available in the recent monograph of G. G. Lorentz et al [4]. However, the general problem of Birkhoff interpolation on the roots of unity has not received any attention. Even in the case of a three row incidence matrix E, we do not know any simple criterion for deciding its regularity on the cube roots of unity. In particular, if E has Hermite sequences of length p and q in the first and third rows and only two nonzero entries in the second row, sufficient conditions for the problem to be regular (or uniquely solvable) on real nodes are known [1, 4]. On the other hand, for a two point interpolation problem the Polya condition is necessary and sufficient for regularity.

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The object of this note is mainly to study three row almost Hermitian matrices. Since the order of the rows is immaterial when the nodes are on the unit circle, we shall say that a matrix is almost Hermitian if all its rows except one are Hermitian. We shall show that a three row almost Hermitian matrix with only two nonzero entries in one row is regular on the cube roots of unity. In Section 2 we give the preliminaries and state our main results for three row almost Hermitian matrices. Section 3 deals with the proof when $p \neq q$.

In Section 4 we consider nodes $1, z_1, \ldots, z_{n-1}$ which correspond to the rows of an almost Hermitian matrix E. We suppose that the z_i's lie in a sector $|\arg(z-1) - \pi| \leqslant \pi/4$ and need not lie on a circle, but are the zeros of a real polynomial. We show that under mild conditions the matrix E is regular on such nodes. Section 5 deals with m iterations of an n-row almost Hermitian matrix. We show that such matrices are regular on mnth roots of unity. As a corollary we obtain an extension of Theorem 1 when $p = q$. In Section 6 we discuss some interpolation problems which are not regular on the roots of unity. Finally, in Section 7, we obtain some sufficient criteria for regularity of almost Hermitian matrices on nth roots of unity.

2. PRELIMINARIES

Let $0 \leqslant k_1 < k_2 < \cdots < k_r$ be r given integers, and let $\{p_i\}_1^{n-1}$ be $n-1$ nonzero integers. We shall denote by $E_n(\{p_i\}_1^{n-1}; k_1, \ldots, k_r)$ the incidence matrix which has $n-1$ Hermitian sequences of length $p_1, \ldots, p_{n-1}$ and one non-Hermitian row with nonzero entries in columns $k_1, \ldots, k_r$. For the sake of brevity we shall denote this matrix by $E_n(\{p_i\}; K(r))$, where $K(r) = \{k_i\}_1^r$, and sometimes by E_n. When $p_1 = p_2 = \cdots = p_{n-1} = p$, we shall denote the matrix by $E_n(p; k_1, \ldots, k_r)$ or $E_n(p; K(r))$.

We shall say that E_n is regular on the nth roots of unity if the interpolation problem defined by E_n is uniquely solvable on the nth roots of unity. We shall suppose that E_n satisfies the strong Polya condition [4]. Without loss of generality we may take the non-Hermitian row in E_n to correspond to any one of the nth roots of unity. It will be convenient to take it to be the first row or the middle row, if n is odd.

In this setup, if the n Hermitian sequences correspond to $\omega, \omega^2, \ldots, \omega^{n-1}$ where $\omega^n = 1$, then we may set

$$\begin{aligned} Q(z) &= \prod_{j=1}^{n-1} (z - \omega^j)^{p_j}, \\ P(z) &= Q(z) \sum_0^{r-1} a_\nu z^\nu. \end{aligned} \tag{2.1}$$

Then the homogeneous interpolation problem on the nth roots of unity given by E_n would require

$$P^{(k_j)}(1)=0, \qquad j=1,2,\ldots,r.$$

Since

$$P^{(k)}(z)=\sum_{0}^{r-1} a_\nu (z^\nu Q(z))^{(k)},$$

we have easily the following

THEOREM A. *A necessary and sufficient condition for E_n to be regular on the nth roots of unity is that E_n satisfies the strong Polya condition and $\Delta_r(E_n)\neq 0$, where $\Delta_r(E_n)$ is a determinant of order r given by*

$$\Delta_r(E_n)=\begin{vmatrix} b_{k_1} & b_{k_1-1} & \cdots & b_{k_1-r+1} \\ b_{k_2} & b_{k_2-1} & \cdots & b_{k_2-r+1} \\ \cdots & \cdots & \cdots & \cdots \\ b_{k_r} & b_{k_r-1} & \cdots & b_{k_r-r+1} \end{vmatrix}, \tag{2.2}$$

in which the b_j's are given by

$$Q(z)=\sum_{\nu=0}^{N} b_j (z-1)^\nu, \qquad N=\sum_{1}^{n-1} p_j. \tag{2.3}$$

In the special case when $r=2$ and $n=3$, Theorem A was given in [1] for real nodes.

REMARK 1. Theorem A can be formulated even in a more general situation. Suppose the Hermitian rows of the length p_i in $E_n(\{p_i\};K(r))$ correspond to the nodes z_i $(i=1,\ldots,n-1)$ and the non-Hermitian row corresponds to 1. If we now set

$$\tilde{Q}(z)=\prod_{1}^{n-1}(z-z_i)^{p_i}, \qquad \tilde{P}(z)=\tilde{Q}(z)\sum_{0}^{r-1} a_\nu z^\nu$$

and if

$$\tilde{Q}(z)=\sum_{0}^{N}\tilde{b}_j(z-1)^j, \qquad N=\sum_{1}^{n-1} p_j,$$

then it follows *mutatis mutandis* that the *necessary and sufficient condition for the regularity of* E_n *is that*

$$\tilde{\Delta}_r(E_n) \neq 0 \tag{2.2a}$$

where $\tilde{\Delta}_r(E_n)$ *is obtained from the determinant* $\Delta_r(E_n)$ *in* (2.2) *by replacing all the entries* $b_{k_i - j}$ *with* $\tilde{b}_{k_i - j}$ $(i = 1, \ldots, r;\ j = 0, 1, \ldots, r-1)$.

REMARK 2. If $r = 1$, and if the z_i's lie on the unit circle, then $\tilde{P}(z) = C\tilde{Q}(z)$ and it follows from the Gauss-Lucas theorem that all the zeros of its derivative lie in the convex hull of the polygon obtained by joining $1, z_1, \ldots, z_{n-1}$. Thus $\tilde{P}^{(k_i)}(1) \neq 0$, which shows that $E_n(\{p_i\}_1^{n-1}; k_1)$ is regular on the unit circle.

We shall devote the next two sections to proving the following when $r = 2$ and $n = 3$:

THEOREM 1. *If* $1 \leqslant p < q$ *are given integers, and if* $E_3(p, q; k_1, k_2)$ *satisfies the strong Polya condition, then it is regular on the cube roots of unity.*

Let 1, ω, ω^2 be the cube roots of unity. Since regularity of any interpolation problem is independent of translation and rotation, we may, by linear transformation, bring 1, ω, ω^2 into 1, $i\sqrt{3}$, and -1 respectively. The strong Polya condition requires that

$$0 < k_1 \leqslant p + q - 1,\ k_1 < k_2 \leqslant p + q. \tag{2.4}$$

In this case (2.2) becomes

$$\Delta_2(E_3) = \begin{vmatrix} Q^{(k_1)}(i\sqrt{3})/k_1! & Q^{(k_1-1)}(i\sqrt{3})/(k_1-1)! \\ Q^{(k_2)}(i\sqrt{3})/k_2! & Q^{(k_2-1)}(i\sqrt{3})/(k_2-1)! \end{vmatrix} \neq 0, \tag{2.5}$$

where $Q(z) = (z-1)^p(z+1)^q$. Since in this case $Q^{(k)}(i\sqrt{3}) \neq 0$, for $k \leqslant p + q$, the condition (2.5) is equivalent to

$$S_{k_1} \neq S_{k_2}, \qquad \text{where} \quad S_k = \frac{1}{k} \cdot \frac{Q^{(k)}(i\sqrt{3})}{Q^{(k-1)}(i\sqrt{3})}.$$

It is easy to see that the following recursion holds:

$$S_k = -\frac{(p+q+2-2k)i\sqrt{3}+q-p}{4k} - \frac{p+q+2-k}{4kS_{k-1}} \qquad (k=2,3,\ldots,p+q), \quad (2.6)$$

$$S_1 = \frac{(p+q)i\sqrt{3}+q-p}{-4}, \qquad S_{p+q+1}=0.$$

TABLE 1

$q=10,\ p=5,\ S_k = a_k + i\sqrt{3}\,b_k,\ k=1,2,\ldots,16$

k	a_k	$\sqrt{3}\,b_k$	k	a_k	$\sqrt{3}\,b_k$
1	(− 1,250000,	− 6.495101)	9	(− 0.065498,	− 0.390231)
2	(− 0.571429,	− 3.092948)	10	(− 0.051792,	− 0.306262)
3	(− 0.349278,	− 1.952465)	11	(− 0.040433,	− 0.236049)
4	(− 0.240365,	− 1.377515)	12	(− 0.030732,	− 0.176124)
5	(− 0.176243,	− 1.028915)	13	(− 0.022195,	− 0.124073)
6	(− 0.134207,	− 0.793600)	14	(− 0.014440,	− 0.078161)
7	(− 0.104583,	− 0.623093)	15	(− 0.007143,	− 0.037115)
8	(− 0.082565,	− 0.493136)	16	(− 0.000000,	− 0.000000)

$q=10,\ p=6,\ k=1,2,\ldots,17$

k	a_k	$\sqrt{3}\,b_k$	k	a_k	$\sqrt{3}\,b_k$
1	(− 1.000000,	− 6.928203)	9	(− 0.055556,	− 0.444747)
2	(− 0.459184,	− 3.313873)	10	(− 0.044690,	− 0.356182)
3	(− 0.282051,	− 2.102147)	11	(− 0.035736,	− 0.282275)
4	(− 0.195139,	− 1.491412)	12	(− 0.028155,	− 0.219339)
5	(− 0.143935,	− 1.121313)	13	(− 0.021563,	− 0.164805)
6	(− 0.110357,	− 0.871690)	14	(− 0.015674,	− 0.116823)
7	(− 0.086700,	− 0.691011)	15	(− 0.010256,	− 0.074019)
8	(− 0.069138,	− 0.553480)	16	(− 0.05102,	− 0.035348)
			17	(− 0.000000,	0.000000)

$q=11,\ p=4,\ k=1,2,\ldots,16$

k	a_k	$\sqrt{3}\,b_k$	k	a_k	$\sqrt{3}\,b_k$
1	(− 1.750000,	− 6.495191)	9	(− 0.091608,	− 0.384271)
2	(− 0.802486,	− 3.083720)	10	(− 0.072271,	− 0.301014)
3	(− 0.491124,	− 1.942048)	11	(− 0.056253,	− 0.231500)
4	(− 0.338058,	− 1.367503)	12	(− 0.042591,	− 0.172286)
5	(− 0.247783,	− 1.019705)	13	(− 0.030598,	− 0.120992)
6	(− 0.188536,	− 0.785260)	14	(− 0.019759,	− 0.075929)
7	(− 0.146755,	− 0.615597)	15	(− 0.009669,	− 0.035885)
8	(− 0.115691,	− 0.486432)	16	(− 0.000000,	− 0.000000)

The numbers $S_k = S_k(p, q)$ can be calculated recursively from (2.6) for different values of k and $p, q \geqslant 1$. A table of some of these numbers is given in Table 1 and is due to the courtesy of Bob Norfolk (Kent State University) and W. Aiello (University of Alberta).

In the light of the above we shall see that Theorem 1 will follow from

THEOREM 2. *If we set* $S_k = a_k + i\sqrt{3}\, b_k$, *where* S_k *is given by* (2.6), *then the numbers* b_k *are negative. Moreover, we have*

$$|b_k| > |b_{k+1}|, \qquad k = 0, 1, \dots, p + q. \tag{2.7}$$

REMARK. From Table 1 of the numbers a_k, b_k, we see that if $p < q$, then all the a_k's are negative and increasing. Also $a_k^2 + 3b_k^2$ decreases as k increases. However, we are not able to prove these in general.

Theorem 1 follows from Theorem 2, since (2.7) implies that $S_{k_1} \neq S_{k_2}$ when $k_1 \neq k_2$ are integers satisfying (2.4).

3. PROOF OF THEOREM 2 (CASE $p \neq q$)

If $p \neq q$, the zeros of $Q^{(k-1)}(z)$ are not symmetric about the origin. Let $p < p$ and let

$$\begin{aligned} \xi^+ &= \{0 \leqslant \xi_1 < \xi_2 < \cdots < \xi_s \leqslant 1\}, \\ \xi^- &= \{-1 \leqslant \xi_t^* < \cdots < \xi_1^* < 0\} \end{aligned} \tag{3.1}$$

denote respectively the nonnegative zeros of $Q^{(k-1)}(z)$ where $s + t = p + q - k + 1$. Then

$$|b_k| = \frac{1}{k}\left[\sum_{j=1}^{s} \frac{1}{\xi_j^2 + 3} + \sum_{j=1}^{t} \frac{1}{\xi_j^{*2} + 3}\right]. \tag{3.2}$$

For the sake of simplicity we may suppose, that the zeros of $Q^{(k-1)}(z)$ are simple. We shall now consider two cases: (i) $\xi_1 = 0$ and (ii) $\xi_1 > 0$.

In case (i), let the zeros of $Q^{(k)}(z)$ be denoted by

$$\begin{aligned}\eta^+ &= \{0<\eta_1<\eta_2<\cdots<\eta_{s-1}\leqslant 1\},\\ \eta^- &= \{-1<\eta_t^*<\eta_{t-1}^*<\cdots<\eta_2^*<\eta_1^*<0\}.\end{aligned} \tag{3.3}$$

We also have the inequalities given by the interlacing property of the zeros of $Q^{(k-1)}(z)$ and $Q^{(k)}(z)$;

$$\begin{aligned}&0=\xi_1<\eta_1<\xi_2<\eta_2<\xi_3<\cdots<\xi_{s-1}<\eta_{s-1}<\xi_s\leqslant 1,\\ &-1\leqslant \xi_t^*<\eta_t^*<\xi_{t-1}^*<\cdots<\xi_1^*<\eta_1^*<0.\end{aligned} \tag{3.4}$$

From (3.4) we have

$$0<\eta_1^{*2}<\xi_1^{*2}<\eta_2^{*2}<\xi_2^{*2}<\cdots<\eta_t^{*2}<\xi_t^{*2}\leqslant 1$$

so that $\eta_j^2>\xi_j^2$ $(j=1,2,\ldots,s-1)$ and $\eta_j^{*2}>\xi_{j-1}^{*2}$ $(j=2,\ldots,t)$. Hence we have

$$\frac{1}{3}+\sum_{j=2}^{s-1}\frac{1}{\xi_j^2+3}>\sum_{j=1}^{s-1}\frac{1}{\eta_j^2+3},$$

$$\sum_{j=1}^{t-1}\frac{1}{\xi_j^{*2}+3}>\sum_{j=2}^{t}\frac{1}{\eta_j^{*2}+3}.$$

Since

$$\frac{1}{\xi_t^{*2}+3}+\frac{1}{\xi_s^2+3}>\frac{2}{4}>\frac{1}{3}>\frac{1}{\eta_1^{*2}+3},$$

it follows that $k|b_k|>(k+1)|b_{k+1}|$, which proves (2.7).

In case (ii), $\xi_1>0$, so that (3.4) is now replaced by

$$\begin{aligned}&0<\xi_1<\eta_1<\xi_2<\cdots<\xi_{s-1}<\eta_{s-1}<\xi_s\leqslant 1,\\ &-1\leqslant \xi_t^*<\eta_t^*<\xi_{t-1}^*<\cdots<\xi_1^*<\eta_1^*<\xi_1.\end{aligned} \tag{3.5}$$

Here η_1^* could be negative, positive, or zero, and the worst situation would be when it is zero.

Since from (3.5) we have

$$0<\eta_1^{*2}<\xi_1^{*2}<\eta_2^{*2}<\cdots<\eta_t^{*2}<\xi_t^{*2}\leqslant 1,$$

it follows that

$$\sum_{j=1}^{s-1}\frac{1}{\xi_j^2+3}>\sum_{j=1}^{s-1}\frac{1}{\eta_j^2+3},\qquad \sum_{j=1}^{t-1}\frac{1}{\xi_j^{*2}+3}>\sum_{j=2}^{t}\frac{1}{\eta_j^{*2}+3},$$

and since we obviously have

$$\frac{1}{\xi_s^2+3}+\frac{1}{\xi_t^{*2}+3}>\frac{1}{3},$$

we have $k|b_k|>(k+1)|b_{k+1}|$, which proves (2.7) in this case.

REMARK 1. When $p=q$, Theorem 2 can be proved in a similar way, or by induction on p, which we outline briefly. Set

$$t_k(p)=\frac{Q_p^{(k)}(i\sqrt{3})}{2ki\sqrt{3}\,Q_p^{(k-1)}(i\sqrt{3})},\qquad Q_p(z)=(z^2-1)^p. \tag{3.6}$$

Since $Q_{p+1}(z)=(z^2-1)Q_p(z)$, we can easily see that

$$t_k(p+1)=t_{k-2}(p)\frac{48t_k(p)t_{k-1}(p)-12t_{k-1}(p)+1}{48t_{k-1}(p)t_{k-2}(p)-12t_{k-2}(p)+1}. \tag{3.7}$$

Also

$$t_k(p)=-\frac{p+1-k}{4k}+\frac{2p+2-k}{48kt_{k-1}(p)}. \tag{3.8}$$

Since $t_1(p)=-p/4$, $t_{2p+1}(p)=0$, $t_{2p}(p)=-1/12p$, it is easy to prove from (3.8) that for every p and $k=1,2,\ldots,2p$, we have $t_k(p)<0$.

We shall show by induction on p that $\{t_k(p)\}$ increases with k. This is clear for $p=1$, and we suppose that

$$t_{k-1}(p)<t_k(p),\qquad k=2,3,\ldots,2p+1. \tag{3.9}$$

We have to show that $t_k(p+1)-t_{k-1}(p+1)>0$ for $k=2,3,\ldots,2p+3$. We shall write t_k for $t_k(p)$ in the following for simplicity. Then for $k \geqslant 4$, we have

$$t_k(p+1)-t_{k-1}(p+1)=N/D,$$

where $D=\{48t_{k-1}t_{k-2}-12t_{k-2}+1\}\{48t_{k-2}t_{k-3}-12t_{k-3}+1\}>0$, since $t_k<0$. Also, after some heavy calculation, one can see that $N=N_1+N_2+N_3$, where N_1, N_2, N_3 are each >0. Indeed, $N_1=48^2t_{k-1}t_{k-2}^2t_{k-3}(t_k-t_{k-1}) + 12\times 48t_{k-1}t_{k-2}t_{k-3}(t_{k-2}-t_k)+48t_{k-1}t_{k-2}(t_k-t_{k-3})>0$, $N_2 = 96t_{k-2}t_{k-3}(t_{k-1}-t_{k-2})+12t_{k-2}(t_{k-3}-t_{k-1})+t_{k-2}-t_{k-3}>0$, and $N_3 = 96t_{k-2}t_{k-3}(t_{k-1}-t_{k-2})>0$. Thus we have proved that $t_k(p+1)-t_{k-1}(p+1)>0$ for $4 \leqslant k \leqslant 2p+1$. When $k=1,2,3$ or $2p+2, 2p+3$, we can easily see from (3.8) that $t_1(p+1)=-(p+1)/4$, $t_2(p+1)=-(p+1)/8+\frac{1}{24}$, $t_3(p+1)=-(p-1)/12-(2p+1)/6(3p+2)$, and $t_{2p+1}(p+1)=-1/2(3p+2)$, $t_{2p+2}(p+1)=-1/12(p+1)$, $t_{2p+3}(p+1)=0$. It can be easily seen that (3.9) is true for $p+1$.

4. MATRIX $E_n(p; k_1, k_2)$ AND SOME GENERAL NODES

Let us consider the almost Hermitian matrix $E_n(p; k_1, k_2)$ with $n-1$ Hermitian sequences each of length p and one non-Hermitian sequence.

We shall now prove a result on interpolation with incidence matrix $E_n(p; k_1, k_2)$, when the nodes are not necessarily on the unit circle, but lie in a sector. More precisely, let $z_1, z_2, \ldots, z_{n-1}$ be the zeros of a *real* polynomial $Q(z)=\sum_0^{n-1} a_\nu z^\nu$, and suppose that all the z_i's lie in the sector $|\arg(z-1)| \geqslant 3\pi/4$, i.e.,

$$|\arg(z_i-1)| \geqslant 3\pi/4, \qquad i=1,2,\ldots,n-1. \tag{4.1}$$

We shall suppose that the first row is non-Hermitian and corresponds to the node 1, and the remaining rows to the nodes $z_1, \ldots, z_{n-1}$. We then prove

THEOREM 3. *Suppose the incidence matrix $E_n(p; k_1, k_2)$ satisfies strong Polya conditions (i.e., $k_1 \leqslant (n-1)p-1$, $k_2 \leqslant (n-1)p$), and suppose the nodes $z_1, \ldots, z_{n-1}$ are the zeros of a real polynomial $Q(z)$ and satisfy* (4.1). *Then $E_n(p; k_1, k_2)$ is regular on the nodes $1, z_1, \ldots, z_{n-1}$.*

Proof. Set $Q_p(z) = [Q(z)]^p$. Then following Remark 1 after Theorem A, it is enough to prove that

$$\Delta = \begin{vmatrix} Q_p^{(k_1)}(1) & k_1 Q_p^{(k_1-1)}(1) \\ Q_p^{(k_2)}(1) & k_2 Q_p^{(k_2-1)}(1) \end{vmatrix} \neq 0. \tag{4.2}$$

We shall in fact show that for any integer k $(1 \leqslant k \leqslant np - p)$, we have

$$\begin{vmatrix} Q_p^{(k)}(1) & Q_p^{(k-1)}(1) \\ Q_p^{(k+1)}(1) & Q_p^{(k)}(1) \end{vmatrix} > 0, \tag{4.3}$$

which implies (4.2), since the hypothesis (4.1) implies that $\operatorname{Re} z_i < +1$, whence $Q_p^{(k)}(1) > 0$, for all k. If

$$Q_p^{(k-1)}(z) = C \prod_{i=1}^{N} (z - \xi_i), \qquad N = np - p - k + 1,$$

we have

$$\frac{Q_p^{(k)}(z)}{Q_p^{(k-1)}(z)} = \sum_{i=1}^{N} \frac{1}{z - \xi_i},$$

so that

$$\frac{Q_p^{(k+1)}(1) Q_p^{(k-1)}(1) - \left[Q_p^{(k)}(1)\right]^2}{\left[Q_p^{(k-1)}(1)\right]^2} = -\sum_{j=1}^{N} \frac{1}{(1 - \xi_j)^2}. \tag{4.4}$$

If $\xi_j = a_j + ib_j$, $j = 1, 2, \ldots, N$, then

$$\sum_{j=1}^{N} \frac{1}{(1 - \xi_j)^2} = \sum_{j=1}^{N} \frac{(1 - a_j)^2 - b_j^2}{\left[(1 - a_j)^2 + b_j^2\right]^2} > 0, \tag{4.5}$$

since the ξ_j's must also lie in the sector $|\arg(z - 1)| \geqslant 3\pi/4$. It follows from (4.5) that

$$Q_p^{(k+1)}(1) Q_p^{(k-1)}(1) < \left[Q_p^{(k)}(1)\right]^2,$$

which yields (4.3). Since $k_1 < k_2$, we get (4.2), which completes the proof. ∎

REMARK. Theorem 3 can be proved even when the p_i's in $E_n(\{p_i\}_1^{n-1}; k_1, k_2)$ are unequal. However, we would require that Hermitian sequences of the same length correspond to conjugate nodes. Thus suppose that $n-1=2s+t$ and that $z_1,\ldots,z_s$ and $\bar{z}_1,\ldots,\bar{z}_s$ are complex and $z_{s+1},\ldots,z_{s+t}$ are real nodes all satisfying (4.1). If the positive integers $p_1,\ldots,p_s$ correspond to $z_1,\ldots,z_s$ and also to $\bar{z}_1,\ldots,\bar{z}_s$ respectively, then we have

$$Q(z)=\prod_{j=1}^{s}(z-z_j)^{p_j}(z-\bar{z}_j)^{p_j}\prod_{j=1}^{t}(z-z_{s+j})^{q_j}.$$

The proof of the above theorem is still valid.

Thus in particular if we consider the incidence matrix $E_4(p,q,p;k_1,k_2)$, then it is regular on the fourth roots of unity.

5. m-ITERATIONS OF $E_n(p;k_1,k_2)$ AND mnth ROOTS OF UNITY

We shall denote by $E_n^{(m)}(p;k_1,k_2)$ the incidence matrix obtained by iterating $E_n(p;k_1,k_2)$ m times in a column. Thus $E_n^{(1)}(p;k_1,k_2)=E_n(p;k_1,k_2)$ and

$$E_n^{(2)}(p;k_1,k_2)=\begin{pmatrix}E_n^{(1)}(p;k_1,k_2)\\ E_n^{(1)}(p;k_1,k_2)\end{pmatrix}$$

and so on. The matrix $E_n^{(m)}(p;k_1,k_2)$ has m rows with two nonzero entries in the kth rows if $k\equiv 1 \pmod{n}$. All the other rows have Hermitian sequences each of length p.

The problem of the regularity of $E_n^{(m)}(p;k_1,k_2)$ on the mnth roots of unity is a special case of a more general problem considered in [7]. There the authors give necessary and sufficient conditions for regularity in a more general case. However, a direct application of their result would require a full explanation of their notation. In view of this we give a direct proof of Theorem 4 formulated after Lemma 2, using the basic tool in [7], which is

LEMMA 1. *If Δ_ν $(\nu=0,1,\ldots,m-1)$ are square matrices each of order $N\times N$ and if $B=(b_{ij})$ is an $m\times m$ matrix, then the matrix Δ given by*

$$\Delta=(b_{ij}\Delta_{j-1}),\qquad 1\leqslant i,j\leqslant m, \tag{5.1}$$

has the determinant

$$|\Delta| = |B|^N \prod_{\nu=0}^{m-1} |\Delta_\nu|.$$

We use this lemma to prove

LEMMA 2. *A necessary and sufficient condition for the regularity of* $E_n^{(m)}(p; k_1, k_2)$ (*satisfying strong Polya condition*) *on* mnth *roots of unity is that*

$$D_\nu\left(E_n^{(m)}\right) \neq 0, \qquad \nu = 0, 1, \ldots, m-1, \tag{5.2}$$

where

$$D_\nu\left(E_n^{(m)}\right) = \begin{vmatrix} R_\nu^{(k_1)}(1) & (z^m R_\nu)_{z=1}^{(k_1)} \\ R_\nu^{(k_2)}(1) & (z^m R_\nu)_{z=1}^{(k_2)} \end{vmatrix} \tag{5.3}$$

and

$$R_\nu(z) = z^\nu \left(\frac{z^{mn} - 1}{z^m - 1} \right)^p, \qquad \nu = 0, 1, \ldots, m-1. \tag{5.4}$$

Proof. Since $\omega^{mn} = 1$, the polynomial $R_0(z)$ is of degree $(n-1)mp$ and satisfies the conditions

$$R_0^{(l)}(\omega^k) = 0, \qquad l = 0, 1, \ldots, p-1, \quad k = 0, 1, \ldots, mn-1, \quad k \not\equiv 0 \pmod{n}. \tag{5.5}$$

Set

$$P(z) = R_0(z) \sum_0^{m-1} c_j z^j + z^m R_0(z) \sum_0^{m-1} c_{j+m} z^j. \tag{5.6}$$

Then from (5.5) it is clear that

$$P^{(l)}(\omega^k) = 0, \qquad l = 0, 1, \ldots, p-1,$$

$$k = 0, 1, \ldots, mn-1, \quad k \not\equiv 0 \pmod{n}.$$

It remains to show that if we require $P(z)$ to satisfy the $2m$ conditions

$$P^{(k_1)}(\omega^j) = P^{(k_2)}(\omega^j) = 0, \qquad j \equiv 0 \pmod{n}, \quad j \leqslant mn - 1, \quad (5.7)$$

then the c_j's in (5.6) are indentically zero. Now the matrix of the system of equations (5.7) can be easily seen to be of the form Δ in (5.1) of Lemma 1 with

$$B = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega^n & \cdots & \omega^{(m-1)n} \\ \cdot & \cdot & \cdots & \cdot \\ 1 & \omega^{(m-1)n} & \cdots & \omega^{2(m-1)n} \end{pmatrix}$$

and

$$\Delta_{j-1} = \begin{pmatrix} R_{j-1}^{(k_1)}(1) & \left(z^m R_{j-1}\right)_{z=1}^{(k_1)} \\ R_{j-1}^{(k_2)}(1) & \left(z^m R_{j-1}\right)_{z=1}^{(k_2)} \end{pmatrix}, \qquad j = 1,2,\ldots,m.$$

Hence from Lemma 1, it follows that

$$|\Delta| = |B|^2 \prod_{j=1}^{m} |\Delta_{j-1}| = |V(1, \omega^n, \ldots, \omega^{(m-1)n})|^2 \prod_{\nu=0}^{m-1} D_\nu\left(E_n^{(m)}\right).$$

Therefore $E_n^{(m)}(p, k_1, k_2)$ is regular on mnth roots of unity if and only if $\prod_0^{m-1} D_\nu(E_n^{(m)}) \neq 0$, from which the result follows. ■

We shall now prove

THEOREM 4. *If the incidence matrix $E_n^{(m)}(p; k_1, k_2)$, $0 \leqslant k_1 \leqslant k_2$, satisfies the strong Polya condition, then it is regular on the mnth roots of unity.*

Proof. We have only to verify (5.2) for all ν, $0 \leqslant \nu \leqslant m-1$. If

$$R_\nu(z) = z^\nu \sum_{j=0}^{np-p} a_j z^{jm},$$

then all the a_j's are positive integers. In fact $R_\nu(z)$ generates a sequence which is PF_2 (Polya frequency of order 2), i.e., $a_j^2 \geqslant a_{j+1} a_{j-1}$, for all j. This follows from the known fact [2] that convolution of PF_2 sequences is a PF_2 sequence and that the coefficients of the polynomial $\sum_{j=0}^{n-1} z^{jm}$ form a PF_2 sequence. Since

$$R_\nu^{(k_1)}(1) = \sum_0^{N-1} a_j(jm+\nu)_{k_1}, \qquad (z^m R_\nu)_{z=1}^{(k_1)} = \sum_0^{N-1} a_j(jm+m+\nu)_{k_1},$$

where $N-1 = np - p$, and $(a)_k = a(a-1)\cdots(a-k+1)$, it follows that

$$D_\nu\left(E_n^{(m)}\right) = \begin{vmatrix} \sum_0^{N-1} a_j(jm+\nu)_{k_1} & \sum_0^{N-1} a_j(jm+m+\nu)_{k_1} \\ \sum_0^{N-1} a_j(jm+\nu)_{k_2} & \sum_0^{N-1} a_j(jm+m+\nu)_{k_2} \end{vmatrix}. \tag{5.8}$$

We now set

$$D_{l,j}(\nu) = \begin{vmatrix} (lm+\nu)_{k_1} & (jm+\nu)_{k_1} \\ (lm+\nu)_{k_2} & (jm+\nu)_{k_2} \end{vmatrix}, \qquad 0 \leqslant j, l \leqslant N-1. \tag{5.9}$$

It is easy to see that

$$D_\nu\left(E_n^{(m)}\right) = \sum_{l=0}^{N-1} \sum_{j=l}^{N-1} \begin{vmatrix} a_l & a_{l-l} \\ a_{j+1} & a_j \end{vmatrix} D_{l,j+1}(\nu), \tag{5.10}$$

where we make the convention that $a_{-1} = a_N = 0$. This also follows from the Cauchy-Binet formula. Since the sequence $a_0, a_1, \ldots, a_{N-1}$ is PF_2, we have

$$\begin{vmatrix} a_l & a_{l-1} \\ a_{j+1} & a_j \end{vmatrix} \geqslant 0, \qquad j \geqslant l.$$

Also we have $D_{l,j+1}(\nu) > 0$ for $j \geqslant l$ and since the strong Polya condition requires that $k_1 \leqslant (n-1)pm - 1 = (N-1)m - 1$ and $k_2 \leqslant Nm - 1$, at least one of the $D_{l,j+1}(\nu)$'s *is* > 0 for $0 \leqslant \nu \leqslant m-1$. In particular $D_{N-1,N}(\nu) > 0$ for $0 \leqslant \nu \leqslant m-1$ and $a_{N-1}^2 - a_N a_{N-2} = 1$. Thus $D_\nu(E_n^{(m)}) > 0$ for $0 \leqslant \nu \leqslant m-1$ which completes the proof.

COROLLARY 1. *If the incidence matrix* $E_n(p; k_1, k_2)$ *satisfies the strong Polya condition i.e.,* $0 \leqslant k_1 \leqslant np - p - 1$ *and* $k_1 < k_2 \leqslant np - p$, *then* $E_n(p; k_1, k_2)$ *is regular on the* nth *roots of unity.*

This generalizes Theorem 1 when $p = q$. Let $\tilde{E}_n(p, q; k_1, k_2)$ *denote an almost Hermitian incidence matrix whose first row is non-Hermitian as above, but whose Hermitian sequences are alternately of length p and q respectively. When* $p = q$, $\tilde{E}_n(p, p; k_1, k_2) = E_n(p; k_1 k_2)$. *Thus*

$$\tilde{E}_n(2,1;2,4) = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let $\tilde{E}_n^{(m)}(p, q; k_1, k_2)$ *denote the incidence matrix obtained by iterating* $\tilde{E}_n(p, q; k_1, k_2)$ *m times in the same way as above. If n is even, the method of Theorem 4 can be used to establish that* $\tilde{E}_n^{(m)}(p, q; k_1, k_2)$ *is regular on the mn th roots of unity if n is even.*

6. MATRIX $E_3(p; \{k_j\}_1^r)$, $r \leqslant 4$

In view of the above it seems natural to ask whether the results of Theorem 1 can be extended to $0 \leqslant k_1 < \cdots < k_r$, $r > 2$. Some numerical evidence for $r = 3$ suggests (see Table 2 at the end of this section) that this is so. However, the methods of Sections 3 and 4 do not seem to work. We recall the concept of k-positive sequences and Polya frequency sequences [3], to obtain sufficient conditions for regularity where $r > 2$.

Given a sequence

$$\ldots, 0, 0, a_0, a_1, \ldots, a_n, 0, 0, \ldots, \qquad a_0 > 0, \quad a_n > 0, \tag{6.1}$$

we consider the matrix

$$A = \|a_{j-1}\| = \begin{pmatrix} a_0 & a_1 & \cdots & a_n & \cdots & 0 & 0 \\ 0 & a_0 & \cdots & a_{n-1} & \cdots & a_n & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & \cdots & a_{n-1} & a_n \end{pmatrix}.$$

We say that the sequence $\{a_j\}$ is a PF_r if all minors

$$A\begin{pmatrix} 1 & 2 & \cdots & k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix} \geqslant 0, \qquad 1 \leqslant j_1 < j_2 < \cdots < j_k, \quad k \leqslant r. \tag{6.2}$$

If strict inequality holds in (6.2), then we say that the sequence (6.1) is a strict PF_r. The polynomial $P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$ is said to generate a PF_r. We mention the following two results [3] in this connection:

(1) If the zeros of the polynomial $P(z)$ are contained in the sector $|\arg z - \pi| \leqslant \pi/(r+1)$, then the sequence $(\ldots,0,0,a_0,a_1,\ldots,a_n,0,0,\ldots)$, with $a_0 > 0, a_n > 0$, is PF_r.

(2) If the sequence $(\ldots,0,0,a_0,a_1,\ldots,a_n,0,0,\ldots) a_n > 0, a_0 > 0$ is PF_r, then the polynomial $a_0 + a_1 z + \cdots + a_n z^n$ has no zeros in the sector $|\arg z| < r\pi/(n+r-1)$.

Thus the sequence $(\ldots,0,0,1,1,1,0,0,\ldots)$ is a PF_2, while the sequence $(\ldots,0,0,1,3,3,0,0,\ldots)$ is a PF_5. Since the determinant of order 6 of the form (6.2) given by

$$\begin{vmatrix} 3 & 3 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 0 & 0 & 0 \\ 0 & 1 & 3 & 3 & 0 & 0 \\ 0 & 0 & 1 & 3 & 3 & 0 \\ 0 & 0 & 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{vmatrix}$$

is negative, the polynomial $z^2 + 3z + 3$ generates a strict PF_5 but not a PF_6. A theorem of Fekete and Polya [2] says that if $R(z)$ and $S(z)$ generate PF_r sequences, then $R(z)S(z)$ also generates a PF_r sequence.

We shall now prove

THEOREM 5. *The interpolation problem given by $E_3(p; \{k_1\}_1^r)$, $r \leqslant 4$, is regular on the cube roots of unity when the matrix satisfies the strong Polya condition.*

The strong Polya condition requires that $k_j \leqslant 2p - 2 + j$, $j = 1,2,3,4$. The condition $r \leqslant 4$ is necessary, as is seen on taking $k_j = j$, $j = 1,2,3,4,5$, and $r = 5$. Then the nontrivial polynomial

$$A_6(x) = (x-1)^6 - (\omega - 1)^6, \qquad \omega^3 = 1,$$

satisfies all the conditions of the homogeneous interpolation problem $E_3(1; \{j\}_1^5)$.

Proof. Since regularity of an interpolation problem is independent of translation, we may consider the nodes to be 0, $-1+\omega$, and $-1+\omega^2$, where $\omega^3 = 1$. Then if

$$Q(z) = (z+1-\omega)^p(z+1-\omega^2)^p = (z^2+3z+3)^p = \sum_0^{2p} a_j z^j, \quad (6.3)$$

then from Theorem A, $E_3(p;\{k_j\}_1^r)$ is regular on the cube roots of unity if and only if $\Delta_r(E_3) \neq 0$, where

$$\Delta_r(E_3) = \begin{vmatrix} a_{k_1} & a_{k_1-1} & \cdots & a_{k_1-r+1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{k_r} & a_{k_r-1} & \cdots & a_{k_r-r+1} \end{vmatrix} \quad (6.4)$$

If $r \leqslant 4$, then $\Delta_r(E_3) \neq 0$, since z^2+3z+3 generates strict PF_4, and the result follows from the observation that $Q(z) = (z^2+3z+3)^p$ also generates a strict PF_4 [2]. ■

Remark. We can obtain a similar result for $E_3(p,p+1;\{k_j\}_1^r)$, $r \leqslant 5$. However, the condition $r \leqslant 5$ is sufficient but not necessary. For if $p=1$, $q=2$, $r=6$, $k_j = j$ $(j=1,\ldots,6)$, then the polynomial

$$p(z) = a_0 + a_1 z^7 + a_2 z^8$$

already satisfies $p^{(k_j)}(0) = 0$, $j = 1,2,\ldots,6$. If we require $p(-1+\omega) = p(-1+w^2) = p'(-1+w^2) = 0$, then we are led to check the determinant

$$\Delta = \begin{vmatrix} 1 & (-1+\omega)^7 & (-1+\omega)^8 \\ 1 & (-1+\omega^2)^7 & (-1+\omega^2)^8 \\ 0 & 7(-1+\omega^2)^6 & 8(-1+\omega^2)^7 \end{vmatrix},$$

which is easily seen to be equal to $-15(\omega-1)^{14} \neq 0$.

We now sketch a proof of

Theorem 6. *The interpolation problem given by $E_3(p,p+1;K(r))$, where $K(r) = \{k_1,k_2,\ldots,k_r\}$, $r \leqslant 5$, is regular on the cube roots of unity provided the strong Polya condition is satisfied.*

Proof. Set $Q_1(z) = Q(z)(z+1-\omega^2) = \sum_{j=0}^{2p+1} b_j z^j$, where $Q(z)$ is given by (6.3). Then

$$b_k = (1-\omega^2)a_k + a_{k-1}, \qquad k = 0, 1, \ldots, 2p+1, \tag{6.5}$$

with the convention that $a_{-1} = a_{2p+1} = 0$. As above, using Theorem A, we need to show that $\Delta_r(E_3) \neq 0$, where

$$\Delta_r(E_3) = \begin{vmatrix} b_{k_1} & b_{k_1-1} & \cdots & b_{k_1-r+1} \\ b_{k_2} & b_{k_2-1} & \cdots & b_{k_2-r+1} \\ \cdots & \cdots & \cdots & \cdots \\ b_{k_r} & b_{k_r-1} & \cdots & b_{k_r-r+1} \end{vmatrix}.$$

On bordering the above determinant by the column

$$\begin{pmatrix} a_{k_1-r} & a_{k_2-r} & \cdots & a_{k_r-r} & 1 \end{pmatrix}^T$$

on the right and on adding a row of zeros after the last row, we get a determinant of order $r+1$. By suitable column operations and on using (6.5), we can easily see that

$$\Delta_r(E_3) = (-1)^r \begin{vmatrix} a_{k_1} & a_{k_1-1} & \cdots & a_{k_1-r} \\ a_{k_2} & a_{k_2-1} & \cdots & a_{k_2-r} \\ \cdots & \cdots & \cdots & \cdots \\ a_{k_r} & a_{k_r-1} & \cdots & a_{k_r-r} \\ 1 & -(1-\omega^2) & \cdots & (-1)^r(1-\omega^2)^r \end{vmatrix}. \tag{6.6}$$

Since $1-\omega^2 = \sqrt{3}\, e^{i\pi/6}$, we can see that

$$\operatorname{Im} \Delta_r(E_3) = \sum_{j=0}^{r} (\sqrt{3})^j \left(\sin \frac{j\pi}{6}\right) A_{r+1,j},$$

where $A_{r+1,j}$ denotes the minor of the jth element of the last row of $\Delta_r(E_3)$ in (6.6). Since $Q(z)$ generates a PF_5 sequence and not all the corresponding determinants $A_{r+1,j}$ are zero, we see that $\operatorname{Im}\Delta_r(E_3)\neq 0$ for $r\leqslant 5$, which completes the proof. ■

REMARK. It is easy to see that if $K(10)=\{j\}_1^5\cup\{j\}_7^{11}$, then the incidence matrix $E_3(1,2;K(10))$ is not regular on the cube roots of unity, since the nontrivial polynomial $z^{12}-(\omega-1)^6z^6$ with $\omega^3=1$ satisfies the homogeneous interpolation problem.

Moreover, if $K(l)=\{j\}_{j=1}^l$, then we can see that $E_3(1,2;K(l))$ is regular on the cube roots of unity, since

$$\begin{vmatrix} 1 & (-1+\omega)^l & (-1+\omega)^{l+1} \\ 1 & (-1+\omega^2)^l & (-1+\omega^2)^{l+1} \\ 0 & l(-1+\omega^2)^{l-1} & (l+1)(-1+\omega^2)^l \end{vmatrix}\neq 0.$$

TABLE 2

1. $p=3,\ q=3,\ E_3(p;k_1,k_2,k_3),\ \Delta_3(E_3)=A+iB$

k_1	k_2	k_3	A	B
2	3	4	0.00000000	0.34280172
2	3	5	0.00000000	0.63749092
2	4	5	0.00000000	0.33478297
3	4	5	0.00000000	0.04009377

2. $p=2,\ q=4,\ E_3(p,q;k_1,k_2,k_3),\ \Delta_3(E_3)=A+iB$

k_1	k_2	k_3	A	B
2	3	4	0.18680927	0.30835779
2	3	5	0.34127236	0.57190914
2	4	5	0.17385630	0.29930848
3	4	5	0.01939321	0.03575713

3. $p=3,\ q=4$

k_1	k_2	k_3	A	B
2	3	4	0.12251060	0.50318066
2	3	5	0.22236465	0.92752448
2	3	6	0.30088172	1.26516380
2	4	5	0.11231800	0.48083877
2	4	6	0.20399115	0.87874154
2	5	6	0.09440429	0.41081470
3	4	5	0.01246397	0.05649495
3	4	6	0.02562006	0.11675840
3	5	6	0.01588723	0.07317538
4	5	6	0.00273114	0.01291194

7. THE DETERMINANT $D(x; k_1,\ldots, k_r)$

Let $\binom{x}{l}$ denote binomial coefficients with the usual convention that

$$\binom{x}{0} = 1, \qquad \binom{x}{l} = 0, \quad l < 0,$$

and

$$\binom{x}{l} = \frac{x(x-1)\cdots(x-l+1)}{l!} \qquad (l = 1,2,\ldots).$$

We shall denote by $D(x; k_1,\ldots, k_r)$ a determinant of order r, whose (i, j) element is the combinatorial number

$$\binom{x}{k_i - j}, \qquad i = 1,\ldots, r, \quad j = 0,1,\ldots, r-1.$$

This kind of determinant occurs in the determination of the regularity of several interpolation problems on roots of unity. An explicit evaluation of this determinant was given by A. M. Ostrowski [4], who shows that it can be used to decide the total positivity of certain sequences. He showed that

$$D(x; k_1,\ldots, k_r) = \frac{\{\Gamma(x+1)\}^r \prod_1^{r-1} (x+\mu)^{r-\mu} V(k_1,\ldots, k_r)}{\prod_{j=1}^{r} \left[\Gamma(k_j+1)\Gamma(x+r-k_j)\right]}, \tag{7.1}$$

where $V(k_1,\ldots, k_r) = \prod_{\lambda > \mu}(k_\lambda - k_\mu)$ is the Vandermondian. We shall suppose that $0 \leqslant k_1 < k_2 < \cdots < k_r$ are integers. On using (7.1) we can sometimes give sufficient conditions for the regularity of $E_n(p; k_1,\ldots, k_r)$ on n roots of unity. We shall prove

THEOREM 7. *If the integers $\{k_i\}_1^r$ $(2 \leqslant r \leqslant (n+1)/2)$ satisfy the inequalities*

$$(n-1)(p-1) + r - 1 \leqslant k_1 < k_2 < \cdots < k_r \leqslant np - p, \tag{7.2}$$

then the incidence matrix $E_n(p; k_1,\ldots, k_r)$ is regular on the nth roots of unity.

Proof. (7.2) implies that the strong Polya condition is satisfied. We shall take the origin to be the point 1, so that the other nodes become $-1+\omega^j$, $j=1,2,\ldots,n-1$, where $\omega^n=1$. Set

$$Q_p(z)=\left(\frac{(z+1)^n-1}{z}\right)^p=\sum_0^{np-p} a_j z^j. \tag{7.3}$$

It follows from Theorem A that the necessary and sufficient condition for the regularity of $E_n(p;k_1,\ldots,k_r)$ on the nth roots of unity is that the following determinant Δ_r is nonzero:

$$\Delta_r=\begin{vmatrix} a_{k_1} & a_{k_1-1} & \cdots & a_{k_1-r+1} \\ a_{k_2} & a_{k_2-1} & \cdots & a_{k_2-r+1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{k_r} & a_{k_r-1} & \cdots & a_{k_r-r+1} \end{vmatrix}. \tag{7.4}$$

Since

$$Q_p(z)=z^{-p}\sum_{j=0}^{p}(-1)^{p-j}\binom{p}{j}(z+1)^{nj},$$

the coefficients of $z^{n(p-1)+1-p},\ldots,z^{np-p}$ are only binomial coefficients, while the coefficients of lower powers of z are some linear combinations of binomial coefficients. More precisely, we have

$$a_j=\binom{np}{j+p}, \qquad (n-1)(p-1)\leqslant j\leqslant np-p.$$

Therefore under the conditions (7.2), we have

$$\Delta_r=D(np;k_1+p,\ldots,k_r+p).$$

From the formula (7.1), we see that $\Delta_r\neq 0$, which proves the result. ■

THEOREM 8. *Suppose the numbers $r,p,n,k_1,\ldots,k_r$ satisfy the following conditions*:

(i) *r is a prime number and* $2<n<r$,
(ii) $p=r^\nu$, $\nu\in Z$, $\nu>0$,
(iii) $0\leqslant k_1<k_2<\cdots<k_r\leqslant(n-1)p+r-1$,
(iv) $k_1,k_2,\ldots,k_r$ *form a complete set of residues modulo* r.

Then the incidence matrix $E_n(p;k_1,\ldots,k_r)$ is regular on the nth *roots of unity.*

REMARK. Condition (iv) is equivalent to the assertion that for $i \neq j$, $k_i \not\equiv k_j \pmod{r}$.

Proof. Since p is a power of a prime by (ii), we have

$$Q_p(z) = \left(\frac{(z+1)^n - 1}{z} \right)^p = \sum_0^{np-p} a_j z^j$$

$$= \left\{ z^{n-1} + \binom{n}{1} z^{n-2} + \cdots + \binom{n}{n-1} \right\}^p$$

$$\equiv z^{(n-1)p} + \binom{n}{1} z^{(n-2)p} + \cdots + \binom{n}{n-1} \pmod{r}.$$

Thus each row of Δ_r defined by (7.4) has precisely one nonzero element modulo r, according to (i) and (iv). (Note that

$$\binom{n}{i} \not\equiv 0 \pmod{r}$$

if $0 \leqslant i \leqslant n$). On the other hand, by (iv) the nonzero elements (mod r) lie in distinct columns. Hence $\Delta_r \not\equiv 0 \pmod{r}$, so that $\Delta_r \neq 0$. ■

EXAMPLE 1. Let $r = 5$, $\nu = 1$ (i.e., $p = 5$), $n = 3$. If $0 \leqslant k_1 < k_2 < \cdots k_5 \leqslant 14$ are integers satisfying (iv), then $E_3(5; k_1, \ldots, k_5)$ is regular on the cube roots of unity. For example, we may take the k_i's to be 1,2,3,4,5 or 0,2,4,6,8.

EXAMPLE 2. Let $r = 5$, $\nu = 2$ (i.e., $p = 25$), $n = 4$. If $0 \leqslant k_1 < k_2 < \cdots < k_5 \leqslant 79$ are integers satisfying (iv), then $E_4(25; k_1, \ldots, k_5)$ is regular on the fourth roots of unity. As examples of k_i's we mention 0,26,52,54,78 and 1, 2,28,29,50.

REMARK. For $j \leqslant (n-1)(p-1)$ the formula (7.1) is not applicable. It would be interesting to know simple necessary and sufficient conditions for regularity of Birkhoff interpolation on roots of unity. The close relationship between interpolation on roots of unity and trigonometric interpolation on equidistant nodes is well known and is highlighted in [7]. Some of the above results have analogues for trigonometric interpolation on a uniform node which are easy to derive.

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